THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 10

1. (a) Find the absolute maximum and minimum values of the function $f(x, y) = xy$ subject the the constraint

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

(b) In fact, the constraint in part (a) defines an ellipse which can be parametrized as $\gamma(t) = (2\sqrt{2}\cos t, \sqrt{2}\sin t)$, where $0 \leq t \leq 2\pi$.

Therefore, the question in part (a) is equivalent to finding absolute extrema of the single variable function $f(\gamma(t))$ (by abuse of notation, it is simply denoted by $f(t)$).

Using techniques in single variable calculus to find absolute extrema of $f(t)$ and verify the answer in (a).

Ans:

(a) Let $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$. Then, we have $\nabla f(x, y) = (y, x)$ and $\nabla g(x, y) = (\frac{x}{4}, y)$.
By the method of Lagrange Multipliers, we let $\nabla f(x, y) = \lambda \nabla g(x, y)$, and so

$$\begin{cases} y = \dfrac{\lambda x}{4} \\ x = \lambda y \\ \dfrac{x^2}{8} + \dfrac{y^2}{2} = 1 \end{cases}$$

Eliminating y from the first two equations, we have $x = \frac{\lambda^2}{4}x$ and so $x = 0$ or $\lambda = \pm 2$. When $x = 0$, $y = \frac{\lambda x}{4} = 0$, but $(x, y) = (0, 0)$ does not satisfiy the last equation; when $\lambda = \pm 2$, $x = \pm 2y$, put them into the last equation, we have $y = \pm 1$.

$f(1, 2) = f(-1, -2) = 2$ and $f(-1, 2) = f(1, -2) = -2$, so $f(x, y)$ attains absolute maximum at $(1, 2)$ and $(-1, -2)$ and it attains absolute minimum at $(-1, 2)$ and $(1, -2)$.

(b) $f(t) = f(\gamma(t)) = (2\sqrt{2}\cos t)(\sqrt{2}\sin t) = 2\sin 2t$.
Then, $f'(t) = 4\cos 2t$. If $f'(t) = 0$, $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
Also $f''(t) = -8\sin 2t$ and $f''(\frac{\pi}{4}) = f''(\frac{5\pi}{4}) = -8 < 0$, $f''(\frac{3\pi}{4}) = f''(\frac{7\pi}{4}) = 8 > 0$. Therefore, $f(t)$ attains maximum at $t = \frac{\pi}{4}, \frac{5\pi}{4}$ and minimum at $t = \frac{3\pi}{4}, \frac{7\pi}{4}$.
We have $f(\frac{\pi}{4}) = f(\frac{5\pi}{4}) = 2$ and $f(\frac{3\pi}{4}) = f(\frac{7\pi}{4}) = -2$.
Therefore, $f(x, y)$ attains absolute maximum at $(1, 2)$ and $(-1, -2)$ and it attains absolute minimum at $(-1, 2)$ and $(1, -2)$.

2. Find the maximum and minimum values of the function $f(x, y, z) = 4x - 7y + 6z$ subjected to the constraint $g(x, y, z) = x^2 + 7y^2 + 12z^2 = 104$.

Ans:

Let $g(x, y, z) = x^2 + 7y^2 + 12z^2 - 104$. Then, we have $\nabla f(x, y, z) = (4, -7, 6)$ and $\nabla g(x, y, z) = (2x, 14y, 24z)$.

By the method of Lagrange Multipliers, we let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, and so

$$\begin{cases} 4 = 2\lambda x \\ -7 = 14\lambda y \\ 6 = 24\lambda z \\ x^2 + 7y^2 + 12z^2 = 104 \end{cases}$$

From the first three equations, we have $x = \frac{2}{\lambda}$, $y = -\frac{1}{2\lambda}$ and $z = \frac{1}{4\lambda}$. Put them into the last equation, we have

$$\begin{aligned} \left(\frac{2}{\lambda}\right)^2 + 7\left(-\frac{1}{2\lambda}\right)^2 + 12\left(\frac{1}{4\lambda}\right)^2 &= 104 \\ \frac{13}{2\lambda^2} &= 104 \\ \lambda &= \pm\frac{1}{4} \end{aligned}$$

When $\lambda = \frac{1}{4}$, $(x, y, z) = (8, -2, 1)$; when $\lambda = -\frac{1}{4}$, $(x, y, z) = (-8, 2, -1)$.
$f(8, -2, 1) = 52$ and $f(-8, 2, -1) = -52$, so $f(x, y, z)$ attains minimum at $(-8, 2, -1)$ and it attains maximum at $(8, -2, 1)$.

3. Let $f(w, x, y, z) = \left(w + \frac{1}{w}\right)^2 + \left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 + \left(z + \frac{1}{z}\right)^2$ for $w, x, y, z > 0$.

   Prove that f is bounded below by $\frac{289}{4}$ on the plane $w + x + y + z = 16$.

   Ans:

   Let $g(w, x, y, z) = w + x + y + z - 16$.

   By the method of Lagrange Multipliers, we let $\nabla f(w, x, y, z) = \lambda \nabla g(w, x, y, z)$, and so

$$\begin{cases} 2\left(w + \frac{1}{w}\right)\left(1 - \frac{1}{w^2}\right) = \lambda \\ 2\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right) = \lambda \\ 2\left(y + \frac{1}{y}\right)\left(1 - \frac{1}{y^2}\right) = \lambda \\ 2\left(z + \frac{1}{z}\right)\left(1 - \frac{1}{z^2}\right) = \lambda \\ w + x + y + z = 16 \end{cases}$$

   From the first two equations, we have

$$\begin{aligned} w - \frac{1}{w^3} &= x - \frac{1}{x^3} \\ w - x &= -\frac{w^3 - x^3}{w^3 x^3} \\ w - x &= -\frac{(w - x)(w^2 + wx + x^2)}{w^3 x^3} \end{aligned}$$

   Therefore, $w = x$ or $1 = -\frac{w^2 + wx + x^2}{w^3 x^3}$, but the second case is rejected since the left hand side is negative. Similarly, we have $w = x = y = z$.

   From the last equation, we have $w = x = y = z = 4$.

   Therefore, f attains an extreme value $f(4, 4, 4, 4) = \frac{289}{4}$.

   The most difficult part is showing that it is indeed the absolute minimum.

   Let $D = \{(w, x, y, z) \in \mathbb{R}^4 : w, x, y, z > 0 \text{ and } w + x + y + z = 16\}$ and in fact, we are finding extreme values of the function $f : D \to \mathbb{R}$ defined by

$$f(w, x, y, z) = \left(w + \frac{1}{w}\right)^2 + \left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 + \left(z + \frac{1}{z}\right)^2.$$

   Let $D' = \{(w, x, y, z) \in \mathbb{R}^4 : w, x, y, z \geq 1/10 \text{ and } w + x + y + z = 16\} \subset D$ which is a compact subset of $\mathbb{R}^4$. Therefore, $f : D' \to \mathbb{R}$ has an absolute minimum by the extreme value theorem.

However, for any point $(w, x, y, z)\partial D'$ of D', there is a coordinate, say $w = 1/10$. Then, at that point, we have

$$f(w, x, y, z) > (w + \frac{1}{w})^2 = (\frac{101}{10})^2 > 100 > \frac{289}{4}.$$

Therefore, the absolute minimum of the function $f : D' \to \mathbb{R}$ attains absolute minimum at $(4, 4, 4, 4)$.

Furthermore, if $(w, x, y, z) \in D \backslash D'$, then there is a coordinate, say w, so that $0 < w < 1/10$. Then, at that point, we have

$$f(w, x, y, z) > (w + \frac{1}{w})^2 > (\frac{1}{w})^2 > 100 > \frac{289}{4}.$$

Therefore, $f(w, x, y, z) \geq \dfrac{289}{4} = f(4, 4, 4, 4)$ for all D.

4. Find the absolute maximum and minimum values of the function $f(x, y, z) = x$ over the curve of intersection of the plane $z = x + y$ and the ellipsoid $x^2 + 2y^2 + 2z^2 = 8$.

Ans:

Let $g_1(x, y, z) = x + y - z$ and $g_2(x, y, z) = x^2 + 2y^2 + 2z^2 - 8$. Then, we are finding extreme values of f on the intersection of $L_0(g_1)$ and $L_0(g_2)$.

By the method of Lagrange multipliers, we let $\nabla f(x, y, z) = \lambda \nabla g_1(x, y, z) + \mu \nabla g_2(x, y, z)$, and so

$$\begin{cases} 1 = \lambda + 2\mu x & --(1) \\ 0 = \lambda + 4\mu y & --(2) \\ 0 = -\lambda + 4\mu z & --(3) \\ 0 = x + y - z & --(4) \\ 0 = x^2 + 2y^2 + 2z^2 - 8 & --(5) \end{cases}$$

From (2) and (3) we have $4\mu(y + z) = 0$. Thus $\mu = 0$ or $y + z = 0$.
Case 1: $\mu = 0$. Then $\lambda = 0$ by (2), and $\lambda = 1$ by (1), so there is no solution for this case.
Case 2: $y + z = 0$. Then $z = -y$ and, by (4), and $x = -2y$. Therefore, by (5), $4y^2 + 2y^2 + 2y^2 = 8$, and so $y = \pm 1$. From this case we obtain two points: $(2, -1, 1)$ and $(-2, 1, -1)$.
The function $f(x, y, z) = x$ has absolute maximum value 2 and absolute minimum value -2 when restricted to the curve $x + y = z$, $x^2 + 2y^2 + 2z^2 = 8$.